On the Infinitude of Twin-Primes

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Abstract

In this article, we create a sieve for twin-primes by using the smallest prime factor criterion and exhibit some of its elementary properties. We offer three proofs for the infinitude of twin-primes: (1) Originally created proof; (2) Euclid's Proof; (3) Pritchard's proof. Finally, we attached the Polignac Conjecture to show its relationship with the Twin-Prime Conjecture, and summarize what is the contribution of this article in mathematics.

1. Introduction

The Twin-Prime Conjecture is one of most elusive unsolved problems in the world. There are many ways to considered a problem. Many attempts have been made to prove this Conjecture using various methods. We seek a way that is simple, suitably fit the problem, and is easy to understand. This is indeed the beauty and power of mathematics.

In this article, we introduce an elementary method for proving the Twin-Prime Conjecture. The outline of this article is the following:

(i) Build a basic set I_0 and its subsets I_k and E_k $(k \in N)$ by using the smallest prime factor criterion. From the elementary properties of the decomposition of I_0 we immediately obtain the characteristics of the pair elements in the basic set I_0, **Theorem 4**:

  (1) $(p_{k+1}, p_{k+1} + 2) \in I_k$, if and only if $(p_{k+1}, p_{k+1} + 2)$ is a pair of twin-prime.

  (2) If $(p_{k+1}, p_{k+1} + 2) \in I_k$, then $(p_{k+1}, p_{k+1} + 2) \in E_{k+1}$ and $(p_{k+1}, p_{k+1} + 2) \notin I_{k+1}$.

  (3) E_{k+1} has no element of a pair of twin-prime, if $(p_{k+1}, p_{k+1} + 2)$ is not a pair of twin-prime.

(ii) Let $t_k = \min I_k$. If each t_k $(k \geq 1)$ is a pair of twin-prime, then proving the Twin-Prime Conjecture will be transformed to proving the sequence $\{t_k\}$ contains an infinite subsequence of twin-primes (**Theorem 14, Proof 1**);

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(iii) If $I_k\{p_{k+1}^2-3\} \neq \phi$, then $I_k\{p_{k+1}^2-3\}$ consists only of pairs of twin-primes in I_k (**Theorem 8**), hence the minimal element t_k of I_k is also a pair of twin-prime (**Theorem 9**). Thus, the kernel of solving the Twin-Prime Conjecture is proving that $I_k\{p_{k+1}^2-3\} \neq \phi$ for all $k \in N$.(**Theorem 13**);

(iv) Theorem 4 leads us to believe the mathematical induction is a suitable method for proving **Theorem 13**: $I_k\{p_{k+1}^2-3\} \neq \phi$, for all $k \in N$.

There are two cases we need to verify $I_k\{p_{k+1}^2-3\} \neq \phi \Longrightarrow I_{k+1}\{p_{k+2}^2-3\} \neq \phi$ if we use the inductive method:

(1) $(p_{k+1}, p_{k+1}+2) \notin I_k\{p_{k+1}^2-3\}$. That **Theorem 4**, and **Lemma 9** of $I_k\{p_{k+1}^2-3\} \subset I_{k+1}\{p_{k+2}^2-3\}$ implies that $I_k\{p_{k+1}^2-3\} \neq \phi \Longrightarrow I_{k+1}\{p_{k+2}^2-3\} \neq \phi$ is true.

(2) $(p_{k+1}, p_{k+1}+2) \in I_k\{p_{k+1}^2-3\}$. By Theorem 4, $(p_{k+1}, p_{k+1}+2) \in E_{k+1}$ and $(p_{k+1}, p_{k+1}+2) \notin I_{k+1}\{p_{k+2}^2-3\}$. Thus, for completing the inductive proof, we just need the following true statement:

$$I_k\{p_{k+1}^2-3\} \neq \phi \Longrightarrow I_k(p_{k+1}^2-3) \geq 2.$$

(v) One benefit of learning the structures of I_k, E_k (**Theorem 5**) is the exact value formula of the counting function $I_k(\pi_k)$ of the set $I_k\{\pi_k\}$ (**Theorem 6**):

$$I_k(\pi_k) = \prod_{i=2}^{k}(p_i-2), \quad k \geq 2.$$

which shows the sequence $\{I_k(\pi_k)\}$ diverges to ∞.

(vi) For using the counting function formula $I_k(\pi_k)$ to prove $I_k\{p_{k+1}^2-3\} \neq \phi$, we created an inequality (**Lemma 3**)

$$p_{k+1} < \pi_s < p_{k+1}^2 - 3 \leq \pi_{s+1},$$

and defined the concepts of $s-$decomposition of a subset A of I_0 and the i^{th} $s-$componet of A. Applying these definitions to the set of $I_k\{\pi_k\}$, we show that the first $s-$ component $I_k\{\pi_s\}$ in the $s-$decomposition of $I_k\{\pi_k\}$ is a subset of $I_k\{p_{k+1}^2-3\}$. Thus, the inequality becomes a bridge between $I_k\{\pi_k\}$ and $I_k\{p_{k+1}^2-3\}$. Thus, if we can show $I_k(\pi_s) \geq 2$ true, then it will imply $I_{k+1}\{p_{k+2}^2-3\} \neq \phi$:

$$I_k\{p_{k+1}^2-3\} \neq \phi \Longrightarrow I_k(\pi_s) \geq 2 \Longrightarrow I_k(p_{k+1}^2-3) \geq 2 \Longrightarrow I_{k+1}(p_{k+2}^2-3) \geq 1.$$

(vii) Introducing the average function of all $s-$components of the $s-$decomposition of $I_k\{\pi_k\}$ and shows it diverges to infinity, which ensures $I_k(\pi_s) \geq 2$ is true for big enough k.

(viii) Base on the construction of above the outline of (i−vii), we completed the original proof for the twin-prime problem (**Theorem 14, Proof 1**).

(ix) Base on Theorem 13, we give another two proofs: **Euclid-type proof** (without Theorem 10) and **Pritchard-type proof** for the infinitude of twin-primes.

(x) A suggested order for reading this article is sections **1, 2, 3, 4, 6, 7, 8, 9, 10, 5, 11.**

2. Definitions and elementary properties

Let N be the natural number set. We say $a,\ b \in N$ are relatively prime if $(a, b) = 1$, where (a, b) represents the greatest common factor of a and b. Let the set of all prime numbers be denoted by $P = \{p_1, p_2, \cdots\}$, where $p_1 = 2,\ p_2 = 3$ and $p_i < p_j$, if $i < j$. We know that the set P is an infinite set. Denote $\pi_k = \prod_{i=1}^{k} p_i$ for $k \geq 1$ and $\pi_0 = 1$.

We construct a basic set:

$$I_0 = \{(\alpha, \alpha + 2) : \alpha \in N\},$$

in which the first coordinate belongs to the natural number set N, and the second coordinate is obtained from adding 2 to the first coordinate.

Notice that $I_0 \neq N \times N$, and I_0 is clearly isomorphic with N.

Definition 1 *For any $k \in N$, define*

$$E_k = \{(\alpha, \alpha + 2) \in I_{k-1} : p(\alpha) = p_k,\ \text{or}\ \ p(\alpha + 2) = p_k\},$$

where $p(\alpha)$ denotes the smallest prime factor of α; and I_k is defined as follows:

$$I_k = \{(\alpha, \alpha + 2) \in I_0 : (\alpha, \pi_k) = (\alpha + 2, \pi_k) = 1\}.$$

E_k is called the k^{th}-*exterior* of I_0 and I_k is called the k^{th}-*interior* of I_0.

It is easy to verify that another equivalent definition of E_k is

$$E'_k = \{(\alpha, \alpha + 2) \in I_0 : \min\{p(\alpha), p(\alpha + 2)\} = p_k\}.$$

In fact, if $(\alpha, \alpha+2) \in E_k$, then $(\alpha, \alpha+2) \in I_{k-1}$, which implies $p(\alpha) \geq p_k$ and $p(\alpha+2) \geq p_k$; $(\alpha, \alpha+2) \in E_k$ and implies $p(\alpha) = p_k$, or $p(\alpha+2) = p_k$, hence $\min\{p(\alpha), p(\alpha+2)\} = p_k$, and $E_k \subset E'_k$. Conversely, if $(\alpha, \alpha+2) \in E'_k$, then $\min\{p(\alpha), p(\alpha+2)\} = p_k$, so $(\alpha, \pi_{k-1}) = (\alpha + 2, \pi_{k-1}) = 1$, hence $(\alpha, \alpha + 2) \in I_{k-1}$ and $p(\alpha) = p_k$, or $p(\alpha + 2) = p_k$, that means $E'_k \subset E_k$. Hence $E_k = E'_k$.

We also define the notation $(a, \pi_k) = 1$ to represent that $a = (\alpha, \alpha + 2)$, and $(\alpha, \pi_k) = (\alpha + 2, \pi_k) = 1$. Thus, $a = (\alpha, \alpha + 2) \in I_k$ if and only if $(a, \pi_k) = 1$.

From the definition of I_k we immediately have the following results.

Theorem 1 *The collection of sets* $\{I_k\}$ *forms a monotone decreasing sequence of sets, i.e.*

$$I_k \supset I_{k+1}, \quad k \in N.$$

□

For ease of discussion we copied here the following lemmas relating to arithmetic progressions (see [**1**]) without proof.

Lemma 1 *Suppose that* $a, d \in N$, $p_k \in P$ *with* $(p_k, d) = 1$. *Let*

$$A = \{a_i : a_i = a + (i-1)d, i \in N\}$$

and

$$a_i = a + (i-1)d, \qquad 1 \le i \le p_k,$$

$$[a_i] = \{a_i + (q-1)dp_k : q \in N\}, \quad 1 \le i \le p_k.$$

Then the collection of sets $\{[a_i] : 1 \le i \le p_k\}$ *is a partition of* A*; that is*
(1) $[a_i] \cap [a_j] = \phi$, *if* $i \ne j$, $1 \le i, j \le p_k$, *and*
(2) $A = \bigcup_{i=1}^{p_k} [a_i]$.

Lemma 2 *For any given* $a, d \in N$, $p_k \in P$ *with* $(p_k, d) = 1$, *there exists a term* a_m, *which is the unique one of the first* p_k *terms of the arithmetic progression*

$$\{a_n : a_n = a + (n-1)d, \quad n \in N\},$$

such that $p_k \mid a_m$.

The following concept and notation are convenient for discussions in the following sections.

Definition 2 *A sequence of pair numbers* $\{(\alpha_i, \beta_i)\}$ *is called a double arithmetic progression with the common difference* d, *if*

$$\alpha_{i+1} - \alpha_i = \beta_{i+1} - \beta_i = d, \text{ for } i \in N.$$

Using the notation $(\alpha, \beta) + d$ to represent $(\alpha + d, \beta + d)$.

We know the definitions of k^{th}-*exterior* and k^{th}-*interior* of I_0, let's now look at the contents of E_1, I_1, E_2, I_2, E_3, and I_3.

First, we know that

$$I_0 = \{(1,3), (2,4), (3,5), (4,6), \cdots\} = \{(n, n+2) : n \in N\}.$$

From the definitions of the exteriors and interiors of I_0, we have

$$\begin{aligned} E_1 &= \{(\alpha, \alpha+2) \in I_0 : p(\alpha) = 2, \text{ or } p(\alpha+2) = 2\} \\ &= \{(2,4),(4,6),(6,8),\cdots\} = \{(2,4) + \pi_1(q-1) : q \in N\}, \end{aligned}$$

$$\begin{aligned} I_1 &= \{(\alpha, \alpha+2) \in I_0 : (\alpha, \pi_1) = (\alpha+2, \pi_1) = 1\} \\ &= \{(1,3),(3,5),(5,7),\cdots\} = \{(1,3) + \pi_1(q-1) : q \in N\}. \end{aligned}$$

We see that $I_0 = E_1 \cup I_1$ and $E_1 \cap I_1 = \phi$.

Notice that $(\pi_1, p_2) = (2,3) = 1$, by Lemma 1, I_1 can be rewritten to be:

$$I_1 = \left\{ \begin{array}{ll} (1,3) + \pi_2(q-1) & \\ (3,5) + \pi_2(q-1) & : \ q \in N \\ (5,7) + \pi_2(q-1) & \end{array} \right\}.$$

From the definitions of E_2 and I_2, we have

$$E_2 = \left\{ \begin{array}{ll} (1,3) + \pi_2(q-1) & \\ & : q \in N \\ (3,5) + \pi_2(q-1) & \end{array} \right\},$$

and

$$I_2 = \left\{ \ (5,7) + \pi_2(q-1) \quad : q \in N \ \right\}.$$

We see that $I_1 = E_2 \cup I_2$ and $E_2 \cap I_2 = \phi$.

Similarly, by Lemma 1, I_2 can be rewritten to be:

$$I_2 = \left\{ \begin{array}{ll} (5,7) + \pi_3(q-1) & \\ (11,13) + \pi_3(q-1) & \\ (17,19) + \pi_3(q-1) & : \ q \in N \\ (23,25) + \pi_3(q-1) & \\ (29,31) + \pi_3(q-1) & \end{array} \right\}.$$

From the definitions of E_3 and I_3, we have

$$E_3 = \left\{ \begin{array}{ll} (5,7) + \pi_3(q-1) & \\ & : q \in N \\ (23,25) + \pi_3(q-1) & \end{array} \right\},$$

$$I_3 = \left\{ \begin{array}{ll} (11,13) + \pi_3(q-1) & \\ (17,19) + \pi_3(q-1) & : \ q \in N \\ (29,31) + \pi_3(q-1) & \end{array} \right\}.$$

We see that $I_2 = E_3 \cup I_3$ and $E_3 \cap I_3 = \phi$.

Next, let's look at the elementary properties between E_k and I_k.

Theorem 2 *There are the following relations between the exteriors and the interiors of I_0. For all $k \in N$,*
(1) $E_k \subset I_{k-1}$,
(2) $I_{k-1} = I_k \cup E_k$,
(3) $I_k \cap E_k = \phi$.

Proof. (1) Follow the definition of E_k.

(2) We just need to verify that $I_{k-1} \subset I_k \cup E_k$. If $(\alpha, \alpha+2) \in I_{k-1}$, then $(\alpha, \pi_{k-1}) = (\alpha+2, \pi_{k-1}) = 1$, so $p(\alpha) \geq p_k$ and $p(\alpha+2) \geq p_k$, in which if one of $p(\alpha) = p_k$ and $p(\alpha+2) = p_k$ is true, then $(\alpha, \alpha+2) \in E_k$. If both $p(\alpha) > p_k$ and $p(\alpha+2) > p_k$, then $(\alpha, \pi_k) = (\alpha+2, \pi_k) = 1$, which implies $(\alpha, \alpha+2) \in I_k$. Hence $I_{k-1} \subset I_k \cup E_k$.

(3) Suppose that $I_k \cap E_k \neq \phi$, then there is an $(\alpha, \alpha+2) \in I_k \cap E_k$, which means that $(\alpha, \pi_k) = (\alpha+2, \pi_k) = 1$ and $\min\{p(\alpha), p(\alpha+2)\} = p_k$. This is impossible. Hence $I_k \cap E_k = \phi$.
□

Theorem 3 *For any $k \in N$, the collection of sets $\{E_1, E_2, \ldots, E_k, I_k\}$ forms a partition of I_0; that is*
(1) $E_i \cap E_j = \phi$, *if* $i \neq j$, $1 \leq i, j \leq k$; *and* $E_i \cap I_k = \phi$, *for all* i, $1 \leq i \leq k$;
(2) $I_0 = E_1 \cup E_2 \cup \cdots \cup E_k \cup I_k$.

Proof. (1) Fix k. If $E_i \cap E_j \neq \phi$, for some i, j with $i \neq j$, then there is an element $(\alpha, \alpha+2) \in E_i \cap E_j$, which implies $p_i = \min\{p(\alpha), p(\alpha+2)\} = p_j$, contradicting to $i \neq j$.

Let $1 \leq i \leq k$, suppose that $E_i \cap I_k \neq \phi$, for some $i, 1 \leq i \leq k$. Let $(\alpha, \alpha+2) \in E_i \cap I_k$, then $\min\{p(\alpha), p(\alpha+2)\} = p_i$ and $(\alpha, \pi_k) = (\alpha+2, \pi_k) = 1$. Since for any $i, 1 \leq i \leq k$, $(p_i, \pi_k) = p_i$, so $(\alpha, \pi_k) \geq p_i$ or $(\alpha+2, \pi_k) \geq p_i$, thus, that $E_i \cap I_k \neq \phi$, for some $i, 1 \leq i \leq k$ is impossible. Hence $E_i \cap I_k = \phi$, for all $1 \leq i \leq k$.

(2) It is trivial to show that $I_0 \supset E_1 \cup E_2 \cup \cdots \cup E_k \cup I_k$. Suppose that $(\alpha, \alpha+2) \in I_0$, if $\min\{p(\alpha), p(\alpha+2)\} = p_i$, for some i, $1 \leq i \leq k$, then $(\alpha, \alpha+2) \in E_i$, or if $\min\{p(\alpha), p(\alpha+2)\} = p_i$, for some i, $i > k$ then $(\alpha, \pi_k) = 1$ and $(\alpha+2, \pi_k) = 1$, so $(\alpha, \alpha+2) \in I_k$, hence $I_0 \subset E_1 \cup E_2 \cup \cdots \cup E_k \cup I_k$. So $I_0 = E_1 \cup E_2 \cup \cdots \cup E_k \cup I_k$ is true.
□

Theorem 4 *Suppose that $k \in N$, then*
(1) $(p_{k+1}, p_{k+1}+2) \in I_k$, *if and only if* $(p_{k+1}, p_{k+1}+2)$ *is a pair of twin-prime.*
(2) *If* $(p_{k+1}, p_{k+1}+2) \in I_k$, *then* $(p_{k+1}, p_{k+1}+2) \in E_{k+1}$ *and* $(p_{k+1}, p_{k+1}+2) \notin I_{k+1}$.
(3) E_{k+1} *has no element which is a pair of twin-prime, if* $(p_{k+1}, p_{k+1}+2)$ *is not a pair of twin-prime.*

Proof. (1) If $(p_{k+1}, p_{k+1}+2) \in I_k$, then $(p_{k+1}+2, \pi_k) = 1$ by the definition of I_k. If $p_{k+1}+2$ is composite, then it can only have a nontrivial prime factor p_{k+1}, but it is impossible besides $2+2 = 2 \times 2$. Hence $(p_{k+1}, p_{k+1}+2)$ is a pair of twin-prime for $k \in N$. If $(p_{k+1}, p_{k+1}+2)$ is a pair of twin-prime, then $(p_{k+1}, \pi_k) = 1$ and $(p_{k+1}+2, \pi_k) = 1$ hence $(p_{k+1}, p_{k+1}+2) \in I_k$.

(2) By Theorem 2(2), $I_k = I_{k+1} \cup E_{k+1}$ and $I_{k+1} \cap E_{k+1} = \phi$. Hence that $(p_{k+1}, p_{k+1}+2) \in I_k$ implies $p(p_{k+1}) = p_{k+1}$ and $(p_{k+1}+2, \pi_k) = 1$ and $p(p_{k+1}+2) \geq p_{k+1}$. Thus, $(p_{k+1}, p_{k+1}+2) \in E_{k+1}$ and adding $E_{k+1} \cap I_{k+1} = \phi$, so $(p_{k+1}, p_{k+1}+2) \notin I_{k+1}$.

(3) Base on (1), (2), and the definition of E_k. □

Theorem 4 responds the following questions.

Question (1). Must (p_k, p_k+2) be in I_{k-1}?

Answer: Yes, if it is a pair of twin-prime, otherwise no, if it is not a pair of twin-prime. For example, $(23, 25) = (p_9, p_9+2) \notin I_{9-1} = I_8$. In fact, $(23, 25) \in E_3$ and $(23, 25) \notin I_3$, since $(25, \pi_3) = 5 \neq 1$. Theorem 1 shows that $I_8 \subset I_3$, hence $(23, 25) \notin I_8 = I_{9-1}$.

Question (2). Must (p_k, p_k+2) or (p_k-2, p_k) be in E_k?

Answer: Yes, if they are twin-primes: $(p_k, p_k+2) \in E_k$ and $(p_k-2, p_k) = (p_{k-1}, p_k) \in E_{k-1}$. Otherwise no, if they are not twin-primes. For example, $p_9 = 23$, then $(p_9, p_9+2) = (23, 25) \in E_3$ and $(p_9-2, p_9) = (21, 23) \in E_2$. Theorem 3 shows $E_3 \cap E_9 = \phi$ and $E_2 \cap E_9 = \phi$, so $(p_9, p_9+2) \notin E_9$ and $(p_9-2, p_9) \notin E_9$.

3. The structures of I_k, E_k

In section 1, we exhibited the contents of E_1, I_1, E_2, I_2, E_3, and I_3. In general, the structures of I_k and E_k are as follows.

Theorem 5 *For $k \geq 2$, the set I_k consists of $\prod_{i=2}^{k}(p_i-2)$ double arithmetic progressions with the common difference π_k, they form a partition of I_k; and for $k \geq 3$, the set E_k consists of $2\prod_{i=2}^{k-1}(p_i-2)$ double arithmetic progressions with a common difference π_k, they form a partition of E_k. Precisely, if $k \geq 2$,*

$$I_k = \bigcup_{i=1}^{\prod_{j=2}^{k}(p_j-2)} [a_i], \tag{1}$$

where

$$a_i = (\alpha_i, \alpha_i+2) \in I_{k-1},\ p_k < \alpha_i \leq \pi_k,\ \text{and } (\alpha_i, \pi_k) = 1, (\alpha_i+2, \pi_k) = 1,$$

and

$$[a_i] = \{(\alpha_i, \alpha_i+2) + \pi_k(q-1) : q \in N\},\ \text{for } 1 \leq i \leq \prod_{j=2}^{k}(p_j-2),$$

are disjoint.

If $k \geq 3$,

$$E_k = \bigcup_{i=1}^{2\prod_{j=2}^{k-1}(p_j-2)} [a_i], \tag{2}$$

where

$$a_i = (\alpha_i, \alpha_i + 2) \in I_{k-1},\ p_{k-1} < \alpha_i \le \pi_k,\ \text{and}\ \ p(\alpha_i) = p_k\ \textit{or}\ p(\alpha_i + 2) = p_k,$$

and

$$[a_i] = \{(\alpha_i, \alpha_i + 2) + \pi_k(q-1) : q \in N\},\ \text{for}\ 1 \le i \le 2\prod_{j=2}^{k-1}(p_j - 2),$$

are disjoint.

Proof. Use the induction on $k \ge 2$ for (1) and $k \ge 3$ for (2).

When $k = 2$,

$$\bigcup_{i=1}^{\prod_{j=2}^{k}(p_j-2)} [a_i] = \bigcup_{i=1}^{\prod_{j=2}^{2}(p_j-2)} [a_i] = \bigcup_{i=1}^{(3-2)} [a_i] = [a_1],$$

where

$$a_i = (\alpha_i, \alpha_i + 2) \in I_1,\ p_2 < \alpha_i \le \pi_2,\ \textit{and}\ (\alpha_i, \pi_2) = 1, (\alpha_i + 2, \pi_2) = 1,$$

and

$$[a_i] = \{(\alpha_i, \alpha_i + 2) + \pi_2(q-1) : q \in N\},\ \text{for}\ 1 \le i \le \prod_{j=2}^{2}(p_j - 2),$$

are disjoint. That is, from $1 \le i \le \prod_{j=2}^{2}(3-2)$, we have $i = 1$; and $3 < \alpha_1 \le 6$, and $(\alpha_1, 6) = 1$, which implies $\alpha_1 = 5$. Thus when $k = 2$,

$$\bigcup_{i=1}^{\prod_{j=2}^{k}(p_j-2)} [a_i] = \bigcup_{i=1}^{\prod_{j=2}^{2}(p_j-2)} [a_i] = \bigcup_{i=1}^{(3-2)} [a_i] = [a_1] = \{(5,7) + \pi_2(q-1) : q \in N\} = I_2.$$

Hence the equality (1) holds when $k = 2$.

Let us look at the equality (2) when $k = 3$.

$$\bigcup_{i=1}^{2\prod_{j=2}^{k-1}(p_j-2)} [a_i] = \bigcup_{i=1}^{2\prod_{j=2}^{3-1}(p_j-2)} [a_i] = \bigcup_{i=1}^{2(p_2-2)} [a_i] = \bigcup_{i=1}^{2}[a_i] = [a_1] \cup [a_2],$$

and

$$[a_i] = \{(\alpha_i, \alpha_i + 2) + \pi_3(q-1) : q \in N\},\ \text{for}\ 1 \le i \le 2\prod_{j=2}^{3-1}(p_j - 2),$$

are disjoint; where $a_i,\ 1 \le i \le 2$ must be satisfied that $a_i = (\alpha_i, \alpha_i + 2) \in I_2$, $3 = p_2 < \alpha_i \le \pi_3 = 30$ and $p(\alpha_i) = p_3 = 5$ or $p(\alpha_i + 2) = p_3 = 5$. Look at the content of I_2, α_i can only be 5, and 23. Hence

$$\bigcup_{i=1}^{2\prod_{j=2}^{3-1}(p_j-2)} [a_i] = [(5,7)] \cup [(23,25)] = E_3.$$

Hence the equality (2) holds when $k = 3$.

Suppose that equality (1) holds for $k \geq 2$ and equality (2) holds for $k \geq 3$. We claim that the equality (1) and (2) hold for $k + 1$.

Notice that the general term of i^{th} equivalent class of I_k is

$$[a_i] = \{(\alpha_i, \alpha_i + 2) + \pi_k(q - 1) : q \in N\},$$

where

$$(\alpha_i, \alpha_i + 2) \in I_{k-1},\ p_k < \alpha_i \leq \pi_k,\ and\ (\alpha_i, \pi_k) = 1,\ (\alpha_i + 2, \pi_k) = 1.$$

Let

$$\beta_t = \alpha_i + \pi_k(t - 1), \quad t = 1, 2, \cdots, p_{k+1}.$$

By Lemma 1,

$$[a_i] = \bigcup_{t=1}^{p_{k+1}} [b_t],$$

where $[b_t] = \{(\beta_t, \beta_t + 2) + \pi_{k+1}(q - 1) : q \in N\},\ 1 \leq t \leq p_{k+1}$ are disjoint. That means each double arithmetic progression $[a_i]$ with the common difference π_k in I_k can be split into p_{k+1} double arithmetic progressions with the common difference π_{k+1}. Since $(p_{k+1}, \pi_k) = 1$, by Lemma 2, there exists a unique term β_i in $\{\beta_1, \cdots, \beta_{p_{k+1}}\}$, such that $p_{k+1} \mid \beta_i$ and $p_{k+1} \not| \beta_t$, if $t \neq i$ and $1 \leq t \leq p_{k+1}$. Similarly, there exists a unique term $\beta_j + 2$ in $\{\beta_1 + 2, \cdots, \beta_{p_{k+1}} + 2\}$, such that $p_{k+1} \mid \beta_j + 2$ and $p_{k+1} \not| \beta_t + 2$ if $t \neq j$ and $1 \leq t \leq p_{k+1}$.

Since $(\beta_i, \beta_i + 2) \in I_k$, $(\beta_i, \pi_k) = (\beta_i + 2, \pi_k) = 1$, adding $p_{k+1} \mid \beta_i$, so $(\beta_i, \beta_i + 2) \in E_{k+1}$. Similarly, $(\beta_j, \beta_j + 2) \in E_{k+1}$.

If $t \neq i, j$, then $p_{k+1} \not| \beta_t$, and $p_{k+1} \not| \beta_t + 2$, are both true, hence $(\beta_t, \pi_{k+1}) = (\beta_t + 2, \pi_{k+1}) = 1$. Hence $(\beta_t, \beta_t + 2) \in I_{k+1}$ for all $t \neq i, j$ and $1 \leq t \leq p_{k+1}$.

Thus, each double arithmetic progression $[a_i] \subset I_k$ can split into p_{k+1} double arithmetic progressions $[b_t]$, $1 \leq t \leq p_{k+1}$, with common difference π_{k+1}, among them there are two progressions belong to E_{k+1} and $p_{k+1} - 2$ progressions belong to I_{k+1}. By the inductive hypothesis, there are $\prod_{i=2}^{k}(p_i - 2)$ progressions of $[a_i]$ in I_k, hence I_{k+1} consists of $(p_{k+1} - 2)\prod_{i=2}^{k}(p_i - 2) = \prod_{i=2}^{k+1}(p_i - 2)$ double arithmetic progressions with the common difference π_{k+1}. Hence the equality (1) holds for all $k \geq 2$. At the same time, we know E_{k+1} consists of $2\prod_{i=2}^{k}(p_i - 2)$ double arithmetic progressions with the common difference π_{k+1}. By the induction, the equality (2) holds for any $k \geq 3$. □

Here we review some concepts and notations as follows:

Definition 3 *Let A be a subset of I_0. Denote $A\{n\} := \{(\alpha, \alpha + 2) \in A : 1 \leq \alpha \leq n\}$. $A\{n\}$ is called the cutout of A by n.*

Definition 4 *Define $A(n)$ be the counting function of the set A:*

$$A(n) = |A\{n\}| = |\{\alpha : (\alpha, \alpha + 2) \in A\{n\}\}|,$$

where $|A|$ is the cardinality of the set A.

Thus, we have

Theorem 6 *Suppose $k \geq 3$, then the values of the counting functions $I_k(n)$, $I_{k-1}(n)$ and $E_k(n)$ at $n = \pi_k$ are respectively*
(1) $I_k(\pi_k) = \prod_{i=2}^{k}(p_i - 2)$,
(2) $E_k(\pi_k) = 2\prod_{i=2}^{k-1}(p_i - 2)$,
(3) $I_{k-1}(\pi_k) = p_k \prod_{i=2}^{k-1}(p_i - 2)$.

Proof. Formulas (1), (2) follow Theorem 5 immediately. For (3), we have,

$$\begin{aligned} I_{k-1}(\pi_k) &= (I_k \cup E_k)(\pi_k) && \text{((2) of Theorem 2)} \\ &= I_k(\pi_k) + E_k(\pi_k) && \text{((3) of Theorem 2)} \\ &= \prod_{i=2}^{k}(p_i - 2) + 2\prod_{i=2}^{k-1}(p_i - 2) && \text{(Formulas (1) and (2))} \\ &= [(p_k - 2) + 2]\prod_{i=2}^{k-1}(p_i - 2) = p_k \prod_{i=2}^{k-1}(p_i - 2). \end{aligned}$$

□

4. About the sequence of $\{t_k\}$

Theorem 7 *For all $k \in N$, $I_k \neq \phi$.*

Proof. When $k = 1, 2$ we know $I_k \neq \phi$ is true. From equation (1) of Theorem 6 we know that when $k \geq 3$, $I_k(\pi_k) = \prod_{i=2}^{k}(p_i - 2) \geq 3 > 0$. Thus, $I_k\{\pi_k\} \neq \phi$, adding $I_k\{\pi_k\} \subset I_k$, which implies $I_k \neq \phi$ for all $k \geq 3$. □

Because I_0 and N are isomorphic, by the *Minimum Principle* (see [**2**, p.3]), a nonempty subset of integers contains its minimal element, hence $\min I_k$ exists and $\min I_k \in I_k$ for all $k \in N$ base on Theorem 7. Denote $t_1 = (3, 5)$ and $t_k = \min I_k,\ k \geq 2$. Then

$$I_k \supset I_{k+1} \ \Rightarrow\ \min I_k \leq \min I_{k+1} \ \Rightarrow\ t_k \leq t_{k+1} \quad \text{for} \quad k \geq 2.$$

Thus, we have an infinite nondecreasing sequence of pairs of the type $t_k = (m_k, m_k + 2)$:

$$\{t_k\}_{k=1}^{\infty}.$$

For more understanding on the structure of I_k we have the following theorem.

Theorem 8 *If $I_k\{p_{k+1}^2 - 3\} \neq \phi$, then $I_k\{p_{k+1}^2 - 3\}$ consists of the pairs of twin-prime only. In particular, $\min I_k\{p_{k+1}^2 - 3\}$ is a pair of twin-prime.*

Proof. Suppose $(\alpha, \alpha+2) \in I_k\{p_{k+1}^2 - 3\}$, then $\alpha \le p_{k+1}^2 - 3$, and $(\alpha, \pi_k) = 1$, $(\alpha + 2, \pi_k) = 1$. Suppose that α is composite, then it is divisible by a prime p such that $p < \sqrt{\alpha} \le \sqrt{p_{k+1}^2 - 3} < p_{k+1}$, that is $p \le p_k$. It contradicts from $(\alpha, \pi_k) = 1$. Suppose that $\alpha + 2$ is composite. We assume that $\alpha + 2 = pq$, $p \le q$, then $p^2 \le pq = \alpha + 2 \le p_{k+1}^2 - 1 < p_{k+1}^2 \Rightarrow p < p_{k+1}$, which implies $p \le p_k$. It also contradicts from $(\alpha+2, \pi_k) = 1$. Hence $I_k\{p_{k+1}^2 - 3\}$ consists of pairs of twin-prime only. $\square$

Theorem 9 *If $I_k\{p_{k+1}^2 - 3\} \ne \phi$, then $t_k = \min I_k$ is a pair of twin-prime for all $k \in N$.*

Proof. $I_k\{p_{k+1}^2 - 3\} \subset I_k \Rightarrow t_k = \min I_k \le \min I_k\{p_{k+1}^2 - 3\}$. Conversely, the set $I_k\{p_{k+1}^2 - 3\} \ne \phi$, so there is an element $(\beta, \beta+2) \in I_k\{p_{k+1}^2 - 3\} \subset I_k$, which implies $\min I_k \le (\beta, \beta+2) < (p_{k+1}^2 - 3, p_{k+1}^2 - 1) \Rightarrow \min I_k \in I_k\{p_{k+1}^2 - 3\}$, so $\min I_k\{p_{k+1}^2 - 3\} \le \min I_k = t_k$. Hence $t_k = \min I_k = \min I_k\{p_{k+1}^2 - 3\}$ and it is a pair of twin-prime by Theorem 8. $\square$

We attached Tables A and B at the end of this article. In Table A we can see the elements in the set of $I_k\{p_{k+1}^2 - 3\}$, its counting number $I_k(p_{k+1}^2 - 3)$, and its minimum element $t_k = \min I_k$, $4 \le k \le 60$. We see that the sequence $\{t_k\}_{k=1}^{\infty}$ is nondecreasing at the last column in table A, but as the number increases we observe some consecutive items of $\{t_k\}_{k=1}^{\infty}$ are repeated. The different I_k can have the same minimum element. For example, $t_5 = t_6$; and $t_{20} = t_{21} = t_{22} = t_{23} = t_{24} = t_{25}$, etc. In Table B we list first 172 pairs of twin-prime and their distribution in I_k, $1 \le k \le 963$. We introduce a new concept for next discussion as follows.

Definition 5 *Suppose that an infinite sequence $\{a_n\}$ is monotone. If there are positive integers k, l such that $a_k \ne a_{k+1}$, $a_{k+l} \ne a_{k+l+1}$ and $a_i = a_{k+1}$, $k+1 \le i \le k+l$, then L=$\{a_i,\ k+1 \le i \le k+l\}$ is called an equal chain of the sequence $\{a_n\}$, and $l = |L|$ is called the length of the equal chain L.*

Thus, from Table B we see that the sequence $\{t_k\}_{k=1}^{\infty}$ is consisted only of equal chains, whose lengths are not less than 2 (when $k \ge 3$). In Table B, we can see that a part $\{t_k\}_{k=120}^{200}$ of the sequence $\{t_k\}_{k=1}^{\infty}$ consists of 12 equal chains.

Theorem 10 *Suppose $\{t_k\}$ consists of the pairs of twin-prime and L is an equal chain in $\{t_k\}$, then the length of L is finite.*

Proof. Assume $t_m = (p, p+2) \in L$ for some $m \in N$. Since the p is a prime, there is a unique integer $n \in N$ such that $p_n = p$. By Theorem 4(2), the pair of twin-prime $(p_n, p_n+2) \in E_n$ and $(p_n, p_n+2) \notin I_n$. Thus, $t_m < t_n$ for all $t_m \in L$. That is, $\max\{m |\ t_m \in L\} \le n - 1$. Hence $|L|$ is finite number. $\square$

5. Pritchard's algorithm for finding all twin-primes. (see [1, 4, 5])

The following important results help us to better understand the structures of $I_k\{\pi_k\}$ and $E_k\{\pi_k\}$.

Theorem 11 *Let* $Z_2 = \{(5,7)\}$. *Define*

$$X_2 = \{c + \pi_2(q-1) : q = 1, 2, \cdots, p_3 \text{ and } c \in Z_2\},$$

$$Y_3 = \{(\alpha, \alpha+2) \in X_2 :\ p(\alpha) = p_3 \text{ or } p(\alpha+2) = p_3\} \ \text{ and } \ Z_3 = X_2 - Y_3.$$

In general, for $k \geq 2$, *define*

$$X_k = \{c + \pi_k(q-1) : q = 1, 2, \cdots, p_{k+1} \text{ and } c \in Z_k\},$$

$$Y_{k+1} = \{(\alpha, \alpha+2) \in X_k :\ p(\alpha) = p_{k+1} \text{ or } p(\alpha+2) = p_{k+1}\} \ \text{ and } \ Z_{k+1} = X_k - Y_{k+1}.$$

Then for each $k \geq 2$, *we have*

(1) $X_k = I_k\{\pi_{k+1}\}$;
(2) $Y_{k+1} = E_{k+1}\{\pi_{k+1}\}$;
(3) $Z_{k+1} = I_{k+1}\{\pi_{k+1}\}$;
(4) $|X_k| = p_{k+1}\prod_{i=2}^{k}(p_i - 2),\ |Y_{k+1}| = 2\prod_{i=2}^{k}(p_i - 2) \text{ and } |Z_{k+1}| = \prod_{i=2}^{k+1}(p_i - 2).$

Proof. Using the induction on $k \geq 2$. First of all, if $k = 2$, then by the definitions of $X_2,\ Y_3$ *and* Z_3 we have

$$X_2 = \{(5,7), (11,13), (17,19), (23,25), (29,31)\},$$

$$Y_3 = \{(5,7), (23,25)\} \ \text{ and } \ Z_3 = \{(11,13), (17,19), (29,31)\}$$

On the other hand, we know that

$$I_2\{\pi_3\} = \{(5,7), (11,13), (17,19), (23,25), (29,31)\},$$

$$E_3\{\pi_3\} = \{(5,7), (23,25)\} \ \text{ and } \ I_3\{\pi_3\} = \{(11,13), (17,19), (29,31)\}.$$

It is clear that (1) – (3) hold for $k = 2$. Through a simple calculation we know (4) holds.

Suppose that (1) – (4) hold for $k \geq 2$.

(1) By the definition we know that

$$X_{k+1} = \{c + \pi_{k+1}(q-1) : q = 1, 2, \cdots, p_{k+2}, \text{ and } c \in Z_{k+1}\}.$$

If $a \in X_{k+1}$, then there exist $c \in Z_{k+1}$ and q, $1 \leq q \leq p_{k+2}$ such that $a = c + \pi_{k+1}(q-1)$. Suppose that $c = (\alpha, \alpha+2)$, then $a = (\alpha + \pi_{k+1}(q-1), \alpha + 2 + \pi_{k+1}(q-1))$. Notice that

$c \in Z_{k+1} = I_{k+1}\{\pi_{k+1}\}$ implies $\alpha < \pi_{k+1}$, which implies $\alpha + \pi_{k+1}(q-1) < \pi_{k+1} + \pi_{k+1}(q-1) = q\pi_{k+1} \le \pi_{k+2}$. Moreover, $(c, \pi_{k+1}) = 1$ implies $(a, \pi_{k+1}) = 1$, hence $a \in I_{k+1}\{\pi_{k+2}\}$. That means

$$X_{k+1} \subseteq I_{k+1}\{\pi_{k+2}\}.$$

On the other hand, from the construction of X_{k+1} and the inductive hypothesis of (4) we know that

$$|X_{k+1}| = p_{k+2} \prod_{i=2}^{k+1} (p_i - 2).$$

By Theorem 6(3),

$$I_{k+1}(\pi_{k+2}) = p_{k+2} \prod_{i=2}^{k+1} (p_i - 2).$$

Hence $X_{k+1} = I_{k+1}\{\pi_{k+2}\}$.

(2) From the definition of Y_{k+2}, $(\alpha, \alpha+2) \in Y_{k+2} \Rightarrow \ (\alpha, \alpha+2) \in X_{k+1} = I_{k+1}\{\pi_{k+2}\}$ and $p(\alpha) = p_{k+2}$ or $p(\alpha+2) = p_{k+2}$, that means $(\alpha, \alpha+2) \in E_{k+2}\{\pi_{k+2}\}$. Hence

$$Y_{k+2} \subseteq E_{k+2}\{\pi_{k+2}\}.$$

On the other hand, from the construction of Y_{k+2} and the inductive hypothesis of (4), we have

$$|Y_{k+2}| = 2 \prod_{i=2}^{k+1} (p_i - 2).$$

Also by Theorem 6(2),

$$E_{k+2}(\pi_{k+2}) = 2 \prod_{i=2}^{k+1} (p_i - 2).$$

Hence $Y_{k+2} = E_{k+2}\{\pi_{k+2}\}$.

(3) From the definition of $Z_{k+2} = X_{k+1} - Y_{k+2}$ and $Y_{k+2} \subseteq X_{k+1}$, we have

$$\begin{aligned} Z_{k+2} &= X_{k+1} - Y_{k+2} = I_{k+1}\{\pi_{k+2}\} - E_{k+2}\{\pi_{k+2}\} \\ &= (I_{k+1} - E_{k+2})\{\pi_{k+2}\} = I_{k+2}\{\pi_{k+2}\}. \end{aligned}$$

(4) In the proofs of (1) and (2), we already proved that

$$|X_{k+1}| = p_{k+2} \prod_{i=2}^{k+1} (p_i - 2) \ \ \textit{and} \ \ |Y_{k+2}| = 2 \prod_{i=2}^{k+1} (p_i - 2).$$

For the last one, Theorem 6(1) implies

$$|Z_{k+2}| = I_{k+2}(\pi_{k+2}) = \prod_{i=2}^{k+2} (p_i - 2).$$

By the induction, $(1) - (4)$ hold for all $k \ge 2$. □

Theorem 12 *Let T presents the set of all pairs of twin-prime in I_0. Suppose $I_k\{p_{k+1}^2 - 2\} \neq \phi$ for all $k \geq 3$.*

(1) *If $\pi_k \leq n < \pi_{k+1}$, for some $k \geq 2$, then*

$$T\{n\} \subset \{t_1, t_2, \cdots, t_{k-1}\} \cup I_k\{n\},$$

where $t_k = \min I_k, \quad k \geq 1$.

(2) *If $p_k^2 - 2 \leq n < p_{k+1}^2 - 2$, $p_k, p_{k+1} \in P$ for some $k \geq 2$, then*

$$T\{n\} = \{t_1, \cdots, t_{k-1}\} \cup I_k\{n\},$$

Proof. (1) It's clear.

(2) If $(\alpha, \alpha+2) \in I_k\{n\}$, then $\alpha \leq n$, and $(\alpha, \pi_k) = 1$, $(\alpha+2, \pi_k) = 1$. If α or $\alpha+2$ is composite, then they are divisible by a prime p such that $p \leq p_k \leq \sqrt{n} < p_{k+1}$, it contradicts from $(\alpha, \pi_k) = 1$, $(\alpha+2, \pi_k) = 1$. Hence $I_k\{n\}$ consists of the pairs of twin-prime only. That is

$$T\{n\} = \{t_1, \cdots, t_{k-1}\} \cup I_k\{n\}.$$

□

Theorem 11 and 12 show an important fact: For any $k \geq 2$ the set

$$\{t_1, \cdots, t_{k-1}\} \cup X_k\{\pi_{k+1}\}$$

covers all pairs of twin-prime in the set $I_0\{\pi_{k+1}\}$. Base on Theorem 11 and 12, we created the Pritchard's algorithm for finding all pairs of twin-prime as follows.

Pritchard's Algorithm Suppose $n > 7$, let $P = \{p_i\}$ be the set of all primes, in which $p_1 = 2$, $p_2 = 3, p_i < p_j$, if $i < j$. Denote $\pi_k = \prod_{i=1}^k p_i$, and $t_1 = (3,5)$. Let $Z_2 = \{(5,7)\}$. and

$$X_2 = \{c + \pi_2(q-1) : q = 1, 2, \cdots, p_3, \text{ and } c \in Z_2\},$$

$$Y_3 = \{(\alpha, \alpha+2) \in X_2 : p(\alpha) = p_3 \text{ or } p(\alpha+2) = p_3\}$$

where $p(\alpha)$ represents the smallest prime factor of α. Let

$$t_2 = \min\ X_2 \quad \text{and} \quad Z_3 = X_2 - Y_3$$

In general, for $k \geq 2$, define

$$X_{k+1} = \{c + \pi_{k+1}(q-1) : q = 1, 2, \cdots, p_{k+2}, \text{ and } c \in Z_{k+1}\},$$

$$Y_{k+2} = \{(\alpha, \alpha+2) \in X_{k+1} : p(\alpha) = p_{k+2} \text{ or } p(\alpha+2) = p_{k+2}\}$$

$$t_{k+1} = \min\ X_{k+1} \quad \text{and} \quad Z_{k+2} = X_{k+1} - Y_{k+2}.$$

If $\pi_r \leq n < \pi_{r+1}$ for some $r \in N$, let $T_r\{n\} = Z_r\{n\}$. Define

$$H_{k+1}\{n\} = \{(\alpha, \alpha+2) \in T_k\{n\} : p(\alpha) = p_k \text{ or } p(\alpha+2) = p_k\}, \quad k \geq r.$$

If $T_k\{n\}\{p_{k+1}^2 - 2\} \neq \phi$, let

$$T_{k+1}\{n\} = T_k\{n\} - H_{k+1}\{n\}, \ \ k \geq r, \ \ \text{and} \ \ t_{k+1} = \min \ T_{k+1}\{n\}.$$

Stop the process at some s where $p_s^2 - 2 \leq n < p_{s+1}^2 - 2$. Let

$$T\{n\} = \{t_1, \cdots, t_s\} \cup T_{s+1}\{n\},$$

where the index of t_s is the same with the index of p_s, then $T\{n\}$ is the set of all pairs of twin-prime in $I_0\{n\}$.

6. An inequality

Lemma 3 *For each $k \geq 10$, there is unique $s \in N$, $s \geq 4$, such that $p_{k+1} < \pi_s < p_{k+1}^2 - 3 \leq \pi_{s+1}$.*

Proof. Let's look at an example for the existence of such k, s. For $k = 10$, we know $p_{11} = 31$, so $p_{11}^2 - 3 = 31^2 - 3 = 958$. $\pi_4 = 210$ and $\pi_5 = 2310$. We see that

$$p_{11} < \pi_4 < p_{11}^2 - 3 \leq \pi_5$$

and $s = 4$. We also see $s = 4$ is unique for $k = 10$. Now, we begin the proof.

Let $k \geq 10$ be fixed and $S = \{i | \pi_i \geq p_{k+1}^2 - 3\}$. First, $S \neq \phi$, since $\lim_{i \to \infty} \pi_i = \infty$. By the Minimum Principle ([**2**, p.3]), $\min S \in S$. Denote $s + 1 = \min S$. The uniqueness of the minimum element $\min S$ implies the uniqueness of s for the k. It's clear that $\pi_{s+1} \geq p_{k+1}^2 - 3 > \pi_s > p_{s+1}^2$, the last part of this inequality is from the *Bonse's inequality* (see [**6**, p.87]): $n \geq 4 \Longrightarrow \pi_n > p_{n+1}^2$. Thus, $p_{k+1} > p_{s+1}$. Hence

$$\pi_s = \frac{\pi_{s+1}}{p_{s+1}} \geq \frac{p_{k+1}^2 - 3}{p_{s+1}} > \frac{(p_{k+1} + 2)(p_{k+1} - 2)}{p_{s+1}} \geq p_{k+1} + 2 > p_{k+1}.$$

□

Notice We can verify immediately that the inequality in Lemma 3 is true when $k = 4, 5, 6,\ s = 3$ and $k = 7, 8, 9,\ s = 4$ respectively.

Lemma 4 *If k, s satisfy the inequality $p_{k+1} < \pi_s < p_{k+1}^2 - 3 \leq \pi_{s+1}$ in Lemma 3, then*

$$\lim_{k \to \infty} s = \infty.$$

Proof. From the inequality $p_{k+1} < \pi_s < p_{k+1}^2 - 3 \leq \pi_{s+1}$, we have

$$k \to \infty \ \ \Longrightarrow \ \ p_{k+1} \to \infty \ \ \Longrightarrow \ \ \pi_s \to \infty \ \ \Longrightarrow \ \ s \to \infty.$$

□

7. The average function $Av(k,s)$

Definition 6 *Suppose a subset $A \subset I_0\{\pi_k\}$, $0 < s < k$. Define $A_i = A \cap [(i-1)\pi_s, i\pi_s]$, $1 \le i \le \prod_{i=s+1}^k p_i$, then*

$$A = \bigcup_{i=1}^{\prod_{i=s+1}^k p_i} A_i$$

is called the s-decomposition of A. Each A_i is called the i^{th} s-component of A.

For example, let $A = I_5\{\pi_5\}$. We see that the 4-decomposition of $I_5\{\pi_5\}$ has eleven 4-components, and the 3-decomposition of $I_5\{\pi_5\}$ has seventy seven 3-components, since $\pi_5 = 11\pi_4 = 77\pi_3$.

From Table B we can easily build Table C, in which the numbers at the column $I_k(\pi_s)$ are the values of the counting function $I_k(\pi_s)$ of the sets $I_k\{\pi_s\}$ for $4 \le k \le 1748$ and $3 \le s \le 9$ and k, s satisfy the inequality in Lemma 3. The numbers in the column of $Av(k,s)$ are the average numbers of the elements in total s-components of the s-decomposition of $I_k\{\pi_k\}$. For example, $\pi_{15} = \prod_{i=5}^{15} p_i \cdot \pi_4$, which implies that the 4-decomposition of $I_{15}\{\pi_{15}\}$ has $\prod_{i=5}^{15} p_i$ 4-components. Using the formula $I_k(\pi_k) = \prod_{i=2}^k (p_i - 2)$ in Theorem 6, the average number of elements in total 4-components of the 4-decomposition of $I_{15}\{\pi_{15}\}$ is

$$\frac{I_{15}(\pi_{15})}{\prod_{i=5}^{15} p_i} = \frac{\prod_{i=2}^{15}(p_i - 2)}{\prod_{i=5}^{15} p_i} \approx 5.35$$

In general, the "*average number*" is defined as follows:

Definition 7 *The average number of s-components in s-decomposition of $I_k\{\pi_k\}$ is defined by*

$$Av(k,s) = \frac{\prod_{i=2}^k (p_i - 2)}{\prod_{i=s+1}^k p_i}.$$

The average function $Av(k,s)$ has the following property:

Lemma 5 *For any positive integers k, s with $k > s$*

(1) $Av(k+1, s) = Av(k,s)\left(1 - \frac{2}{p_{k+1}}\right)$;

(2) $Av(k+1, s+1) = Av(k,s)\left(1 - \frac{2}{p_{k+1}}\right) p_{s+1}$.

Proof.

$$(1) \quad Av(k+1,s) = \frac{\prod_{i=2}^{k+1}(p_i - 2)}{\prod_{i=s+1}^{k+1} p_i} = Av(k,s) \cdot \frac{p_{k+1} - 2}{p_{k+1}} = Av(k,s)\left(1 - \frac{2}{p_{k+1}}\right).$$

$$(2) \quad Av(k+1, s+1) = \frac{\prod_{i=2}^{k+1}(p_i - 2)}{\prod_{i=s+2}^{k+1} p_i} = \frac{\prod_{i=2}^{k}(p_i - 2)(p_{k+1} - 2)}{\prod_{i=s+1}^{k} p_i \cdot \frac{p_{k+1}}{p_{s+1}}} = Av(k,s)\left(1 - \frac{2}{p_{k+1}}\right) p_{s+1}.$$

□

Define

$$k_s = \min\{k | p_{k+1} < \pi_s < p_{k+1}^2 - 3 < \pi_{s+1}\}$$

and

$$K_s = \max\{k | p_{k+1} < \pi_s < p_{k+1}^2 - 3 < \pi_{s+1}\}.$$

It's clear that $K_s + 1 = k_{s+1}$ and $K_s > k_s > s$.

Denote: $l_s = K_s - k_s$.

By Lemma 5(1), the sequence $\{Av(k, s)\}$ is decreasing on k when s is fixed. More precisely,

Corollary 1 *Suppose that k, s satisfy $p_{k+1} < \pi_s < p_{k+1}^2 - 3 \leq \pi_{s+1}$, then*

$$Av(k_s + 1, s) = Av(k_s, s)\left(1 - \frac{2}{p_{k_s+1}}\right),$$

and in general

$$Av(k_s + l, s) = Av(k_s, s)\prod_{i=1}^{l}\left(1 - \frac{2}{p_{k_s+i}}\right), \quad 1 \leq l \leq l_s.$$

Proof. Following Lemma 5(1) immediately. □

Corollary 1 offered a formula, which shows how to count the values of $Av(k_s + l, s)$ base on the value of $Av(k_s, s)$ for all $1 \leq l \leq l_s$ when s is fixed.

Corollary 2 *Suppose that k, s are satisfied $p_{k+1} < \pi_s < p_{k+1}^2 - 3 \leq \pi_{s+1}$. Then*

$$Av(K_{s+1}, s + 1) = Av(K_s, s) \cdot p_{s+1}\prod_{i=0}^{l_{s+1}}\left(1 - \frac{2}{p_{k_{s+1}+i}}\right)$$

where $l_{s+1} = K_{s+1} - k_{s+1}$.

Proof.

$$\begin{aligned}
Av(K_{s+1}, s+1) &= Av(k_{s+1} + l_{s+1}, s+1) \\
&= Av(k_{s+1}, s+1)\prod_{i=1}^{l_{s+1}}\left(1 - \frac{2}{p_{k_{s+1}+i}}\right) && \text{(Corollary 1)} \\
&= Av(K_s, s)\left(1 - \frac{2}{p_{k_{s+1}}}\right) \cdot p_{s+1} \cdot \prod_{i=1}^{l_{s+1}}\left(1 - \frac{2}{p_{k_{s+1}+i}}\right) && \text{(Lemma 5(2))} \\
&= Av(K_s, s) \cdot p_{s+1} \cdot \prod_{i=0}^{l_{s+1}}\left(1 - \frac{2}{p_{k_{s+1}+i}}\right).
\end{aligned}$$

□

Corollary 3 *Suppose that integers* $k,\ s$ *satisfy the inequality* $p_{k+1} < \pi_s < p_{k+1}^2 - 3 \le \pi_{s+1}$, *then for all* $s \ge 3$,

$$Av(K_s, s) > 0.$$

Proof. From table C, we have $Av(K_3, 3) = 1.75325 > 0$. The conclusions follow the induction and the formula in Corollary 2. □

Corollary 4 *Suppose that integers* $k,\ s$ *satisfied the inequality* $p_{k+1} < \pi_s < p_{k+1}^2 - 3 \le \pi_{s+1}$. *Then*

$$\lim_{s\to\infty} \frac{Av(k_{s+1}, s+1)}{Av(K_s, s)p_{s+1}} = 1.$$

Proof. Notice $K_s + 1 = k_{s+1}$ and from Lemma 5(2) we have

$$Av(k_{s+1}, s+1) = Av(K_s + 1, s+1) = Av(K_s, s)\left(1 - \frac{2}{p_{k_{s+1}}}\right) p_{s+1}.$$

Corollary 3 ensures $Av(K_s, s) > 0$ for all integer s. Hence

$$\lim_{s\to\infty} \frac{Av(k_{s+1}, s+1)}{Av(K_s, s)p_{s+1}} = 1.$$

□

Corollary 5 *Suppose that* k, s *satisfy the inequality* $p_{k+1} < \pi_s < p_{k+1}^2 - 3 \le \pi_{s+1}$, *then the sequence*

$$\left\{\{Av(k, s)\}_{k=k_s}^{K_s}\right\}_{s=3}^{\infty}$$

diverges to infinity when $k \to \infty$.

Proof. Corollary 3 shows $Av(K_s, s) > 0$. In fact we see $\min Av(K_s, s) \ge 1.75325$ in table C. Thus, $Av(k_{s+1}, s+1) \sim Av(K_s, s)p_{s+1}$ in Corollary 4 ensures that the subsequence $\{Av(k_s, s)\}_{s=3}^{\infty}$ of $\left\{\{Av(k, s)\}_{k=k_s}^{K_s}\right\}_{s=3}^{\infty}$ diverges to ∞. Finally

$$\left\{\{Av(k, s)\}_{k=k_s}^{K_s}\right\}_{s=3}^{\infty}$$

diverges to infinity. □

8. The counting function $I_k(\pi_s)$ of the set $I_k\{\pi_s\}$

Lemma 6 *Suppose that* k, s *are satisfied* $p_{k+1} < \pi_s < p_{k+1}^2 - 3 \le \pi_{s+1}$ *and* $I_k\{\pi_s\} \ne \phi$. *Then sequence* $\{I_k(\pi_s)\}$ *is non-increasing on* k *when* s *is fixed. Also sequence* $\{I_k(\pi_s)\}$ *decreases not more than one when* k *increases one and* s *is fixed* :

$$I_k(\pi_s) - 1 \le I_{k+1}(\pi_s) \le I_k(\pi_s).$$

Proof. First, $I_{k+1}\{\pi_s\} \subset I_k\{\pi_s\}$, since $I_{k+1} \subset I_k$ by Theorem 1. Hence $I_{k+1}(\pi_s) \leq I_k(\pi_s)$. On the other hand, $\pi_s < p_{k+1}^2 - 3$ implies $I_k\{\pi_s\} \subset I_k\{p_{k+1}^2 - 3\}$. Thus, $I_k\{\pi_s\} \neq \phi$ implies $I_k\{p_{k+1}^2 - 3\} \neq \phi$. Hence $I_k\{\pi_s\}$ consists only of twin-primes by Theorem 8. Thus, $I_k\{\pi_s\}$ may lose one element in the only case if $t_k = (p_{k+1}, p_{k+1}+2)$, since $(p_{k+1}, p_{k+1}+2) \in E_{k+1}$ and $(p_{k+1}, p_{k+1}+2) \notin I_{k+1}$ by Theorem 4(2). Hence

$$I_k(\pi_s) - 1 \leq I_{k+1}(\pi_s)$$

□

Denote the maximal decreasing amplitude of $I_k(\pi_s)$ for the fixed s to be:

$$c_s = I_{k_s}(\pi_s) - I_{K_s}(\pi_s).$$

Corollary 6 *Suppose that k, s satisfy $p_{k+1} < \pi_s < p_{k+1}^2 - 3 \leq \pi_{s+1}$ and $I_{k_s}\{\pi_s\} \neq \phi$, then $c_s + 1$ is the number of equal chains of $I_k\{\pi_s\}$ in the interval $[k_s, K_s]$.*

Proof. Follows Lemma 6. □

For example, from Table C, we have

$$c_5 = I_{k_5}(\pi_5) - I_{K_5}(\pi_5) = I_{15}(\pi_5) - I_{39}(\pi_5) = 64 - 58 = 6.$$

and

$$c_6 = I_{k_6}(\pi_6) - I_{K_6}(\pi_6) = I_{40}(\pi_6) - I_{126}(\pi_6) = 456 - 438 = 18.$$

Base on table C, we observe that there are exactly 7(=c_5+1) equal chains of $I_{39}\{\pi_5\}$ in the interval $[k_5, K_5]$=[15,39] and exact 19(=c_6+1) equal chains of $I_{126}\{\pi_6\}$ in the interval $[k_6, K_6]$=[40,126] respectively.

In general, we have $c_s < \frac{1}{2}(K_s - k_s) + 1$, since the length of shortest equal chain is 2.

Corollary 7 *Suppose that k, s satisfy $p_{k+1} < \pi_s < p_{k+1}^2 - 3 \leq \pi_{s+1}$ and $I_k\{\pi_s\} \neq \phi$. Then the sequence*

$$\left\{\{I_s(\pi_s)\}_{k=k_s}^{K_s}\right\}_{s=3}^{\infty}$$

is the upper bound of the sequence $\left\{\{I_k(\pi_s)\}_{k=k_s}^{K_s}\right\}_{s=3}^{\infty}$. Moreover, it diverges and its limit is the same with

$$\lim_{s\to\infty} I_s(\pi_s) = \infty.$$

Proof. By Lemma 6 and the following inequality,

$$I_s(\pi_s) \geq I_k(\pi_s)$$

where $k_s \leq k \leq K_s$. It diverges to infinity is clear since

$$\lim_{s\to\infty} I_s(\pi_s) = \lim_{s\to\infty} \prod_{i=2}^{s} (p_i - 2) = \infty.$$

□

Corollary 7 shows the upper bound sequence $\left\{\{I_s(\pi_s)\}_{k=k_s}^{K_s}\right\}_{s=3}^{\infty}$ of the sequence

$$\left\{\{I_k(\pi_s)\}_{k=k_s}^{K_s}\right\}_{s=3}^{\infty}$$

diverges to infinity, and in Table C, we see the sequence $\left\{\{I_k(\pi_s)\}_{k=k_s}^{K_s}\right\}_{s=3}^{\infty}$ also seems trending to infinity. If we can find a positive lower bound sequence of $\left\{\{I_k(\pi_s)\}_{k=k_s}^{K_s}\right\}_{s=3}^{\infty}$, for example $\left\{\{p_s\}_{k=k_s}^{K_s}\right\}_{s=4}^{\infty}$, which diverges to infinity, or prove $I_k\{\pi_s\} \neq \phi$ for all $k_s \leq k \leq K_s,\ s \geq 3$, then Theorem 13 would be true immediately. But these stronger conditions require more work. Fortunately, the following weaker result (Lemma 7) is sufficient to help us complete the remarkable inductive proof of the infinitude of twin-primes.

Lemma 7 *Suppose integers k, s satisfy the inequalities $p_{k+1} < \pi_s < p_{k+1}^2 - 3 \leq \pi_{s+1}$, then $I_k(\pi_s) \geq 2$, if $I_k\{\pi_s\} \neq \phi$ when k, s are large enough and where $k_s \leq k \leq K_s,\ s \geq 3$.*

Proof. $Av(k, s)$ is the average number of all s-components in s-decomposition of $I_k\{\pi_k\}$ and $I_k(\pi_s)$ is the number of the first s-component of the s-decomposition of $I_k\{\pi_k\}$. Base on the general view of the approximation theory, we can naturally estimate the number of $I_k(\pi_s)$ by the average number $Av(k, s)$, if the set $I_k\{\pi_s\}$ is not empty. Hence

$$I_k(\pi_s) \sim Av(k, s)$$

if $I_k\{\pi_s\} \neq \phi$. Corollary 5 shows that the sequence

$$\left\{\{Av(k, s)\}_{k=k_s}^{K_s}\right\}_{s=3}^{\infty}$$

diverges to infinity. Hence $I_k(\pi_s) \geq 2$, if $I_k\{\pi_s\} \neq \phi$ when k, s are big enough. □

From Table C, in fact, the conclusion of Lemma 7 was true for all k, s with $k_s \leq k \leq K_s,\ s \geq 3$ and k, s satisfy the inequality in Lemma 3.

9. The true statement of $I_k\{p_{k+1}^2 - 3\} \neq \phi$ for all $k \in N$

Lemma 8 *If $I_k\{p_{k+1}^2 - 3\} \neq \phi$, and $t_k = \min I_k$, then $I_k\{p_{k+1}^2 - 3\} - \{t_k\} \subset I_{k+1}\{p_{k+2}^2 - 3\}$.*

Proof. Suppose $I_k\{p_{k+1}^2 - 3\} - \{t_k\} \neq \phi$ and $(\alpha, \alpha + 2) \in I_k\{p_{k+1}^2 - 3\} - \{t_k\}$, then $(\alpha, \alpha+2)$ is a pair of twin-prime by Theorem 8; $\alpha > p_{k+1}$ by Theorem 4(1); $(\alpha, \alpha+2) \in I_k$ implies $(\alpha, \pi_k) = (\alpha + 2, \pi_k) = 1$, adding $\alpha > p_{k+1}$ imply $(\alpha, \pi_{k+1}) = (\alpha + 2, \pi_{k+1}) = 1$, so $(\alpha, \alpha + 2) \in I_{k+1}$. $\alpha \leq p_{k+1}^2 - 3 < p_{k+2}^2 - 3$ implies $(\alpha, \alpha + 2) \in I_{k+1}\{p_{k+2}^2 - 3\}$. Hence $I_k\{p_{k+1}^2 - 3\} - \{t_k\} \subset I_{k+1}\{p_{k+2}^2 - 3\}$. □

Lemma 9 *If $I_k\{p_{k+1}^2 - 3\} \neq \phi$, and $t_k = \min I_k = (p_n, p_n + 2),\ n > k + 1$, then $I_k\{p_{k+1}^2 - 3\} \subset I_{k+1}\{p_{k+2}^2 - 3\}$.*

Proof. $I_k\{p_{k+1}^2-3\} \neq \phi$ makes $(p_n, p_n+2) = \min I_k \in I_k\{p_{k+1}^2-3\}$, so (p_n, p_n+2) is a pair of twin-prime by Theorem 8. $n > k+1$ implies $(p_n, \pi_{k+1}) = (p_n+2, \pi_{k+1}) = 1$, so $(p_n, p_n+2) \in I_{k+1}\{p_{k+1}^2-3\} \subset I_{k+1}\{p_{k+2}^2-3\}$. Thus, from Lemma 8

$$I_k\{p_{k+1}^2-3\} = (I_k\{p_{k+1}^2-3\} - \{t_k\}) \cup \{(p_n, p_n+2)\} \subset I_{k+1}\{p_{k+2}^2-3\}.$$

□

Theorem 13 *$I_k\{p_{k+1}^2-3\} \neq \phi$, for all $k \in N$.*

Proof. From Table C, we see that $I_k\{p_{k+1}^2-3\} \neq \phi$ is true for $1 \leq k \leq 1749$, since $I_k(p_{k+1}^2-3) \geq I_k(\pi_s)$, where k and s satisfy the inequalities $p_{k+1} < \pi_s < p_{k+1}^2-3 \leq \pi_{s+1}$.

Suppose $I_k\{p_{k+1}^2-3\} \neq \phi$ is true for a big enough integer where k with s satisfy the inequalities $p_{k+1} < \pi_s < p_{k+1}^2-3 \leq \pi_{s+1}$, we claim that $I_{k+1}\{p_{k+2}^2-3\} \neq \phi$ is also true.

(1) Suppose that $\min I_k = (p_n, p_n+2)$ and $n > k+1$. By Lemma 9, $I_k\{p_{k+1}^2-3\} \subset I_{k+1}\{p_{k+2}^2-3\}$. Thus, $I_k\{p_{k+1}^2-3\} \neq \phi$ implies $I_{k+1}\{p_{k+2}^2-3\} \neq \phi$.

(2) Suppose that $\min I_k = (p_{k+1}, p_{k+1}+2)$. Then there is a unique integer s such that k and s satisfy the inequalities $p_{k+1} < \pi_s < p_{k+1}^2-3 \leq \pi_{s+1}$. Thus, $(p_{k+1}, p_{k+1}+2) \in I_k\{\pi_s\}$, since $p_{k+1} < \pi_s$. Hence $I_k\{\pi_s\} \neq \phi$. By Lemma 7, $I_k(\pi_s) \geq 2$. By Lemma 8 and $\pi_s < p_{k+1}^2-3$ we have

$$I_{k+1}(p_{k+2}^2-3) \geq I_k(p_{k+1}^2-3) - 1 \geq I_k(\pi_s) - 1 \geq 2-1 = 1$$

That means $I_{k+1}\{p_{k+2}^2-3\} \neq \phi$. By the induction, we know that $I_k\{p_{k+1}^2-3\} \neq \phi$, for all $k \geq 1$. □

Corollary 8 *$t_k = \min I_k$ is a pair of twin-prime for all $k \in N$.*

Proof. Combining Theorem 9 and Theorem 13 we at once reach this conclusion. □

10. Three proofs for the infinitude of twin-primes

We offer three proofs for the infinitude of twin-primes as follows.

Theorem 14 *There is an infinite number of twin-primes.*

Proof 1. Combining Theorem 10 and Corollary 8 we can immediately conclude that the sequence $\{t_k\}_{k=1}^{\infty}$ contains an infinite number of twin-primes.

Proof 2. (Euclid's Proof)

(1) Let $W = \{k \in N |\ (p_{k+1}, p_{k+1}+2) \in I_k\}$. If $k \in W$ then $(p_{k+1}, p_{k+1}+2) \in I_k$, hence $(p_{k+1}, p_{k+1}+2)$ is a pair of twin-prime. We claim that W is an infinite set.

First, $2 \in W$ implies $W \neq \phi$. Suppose that W is finite. let $\kappa = \max_{k\in W} k+1$, then $\kappa < \infty$. Notice that $I_\kappa\{p_{\kappa+1}^2-3\} \neq \phi$ by Theorem 13.

Suppose $(\alpha, \alpha+2) \in I_\kappa\{p_{\kappa+1}^2 - 3\}$. By Theorem 8, $(\alpha, \alpha+2)$ is a pair of twin-prime, hence there is a unique $n \in N$ such that $p_{n+1} \in P$, and $(p_{n+1}, p_{n+1}+2) = (\alpha, \alpha+2)$, which is an element of twin-primes in I_κ. $(p_{n+1}, \pi_\kappa) = (p_{n+1}+2, \pi_\kappa) = 1$ implies $n+1 > \kappa$. Thus, $n > \max_{k\in W} k$, which implies $n \notin W$. On the other hand, $(p_{n+1}, p_{n+1}+2)$ is a pair of twin-prime implies $(p_{n+1}, p_{n+1}+2) \in I_n$ by Theorem 4(1), thus, $n \in W$. Contradiction! Hence W is an infinite set.

(2) Let $T = \{(p_{k+1}, p_{k+1}+2) | k \in W\}$. Then $W \neq \phi$ implies $T \neq \phi$. Suppose that $k_1, k_2 \in W$, and $k_1 \neq k_2$ then $p_{k_1+1} \neq p_{k_2+1}$, and $(p_{k_1+1}, p_{k_1+1}+2) \neq (p_{k_2+1}, p_{k_2+1}+2)$. Thus, W is infinite set implies T is infinite set. Hence there is an infinite number of twin-primes. □

Proof 3. (Pritchard's proof)

In Theorem 11, we obtain

$$|Z_{k+1}| = \prod_{i=2}^{k+1}(p_i - 2) > 0,$$

which means the wheel sieve can be running forever. Let $t_k = \min X_k$. We have an infinite sequence $\{t_k\}$. Corollary 8 shows each element of $\{t_k\}_{k=1}^\infty$ is a pair of twin-prime. From Theorem 10, we know $\{t_k\}_{k=1}^\infty$ contains an infinite number of twin-primes. □

11. The Polignac Conjecture

Finally, we introduce the Polignac's Conjecture. Let

$$T_n = \{(p,q) \mid p - q = 2n, \quad p, q \in P\} \qquad n \in N.$$

For examples,

$$T_1 = \{(3,5), (5,7), (11,13), (17,19), (29,31), (41,43), \cdots\},$$
$$T_2 = \{(3,7), (7,11), (13,17), (19,23), (37,41), (43,47), \cdots\},$$
$$T_3 = \{(5,11), (7,13), (11,17), (13,19), (17,23), (23,29), \cdots\},$$

and so on.

In 1849, Polignac in his article (see [**3**]) conjectured that T_n is an infinite set for all natural number n. We see that the Twin-Prime Conjecture is a special case of the Polignac Conjecture when $n = 1$.

12. Conclusion

Finally here are the contributions of this article to number theory in mathematics:

1. It presented a sieve for the set of all twin-primes by using the smallest prime factor criterion;

2. It exhibited the distribution of twin-primes in the integer set N in Tables A, B, and C for the first time;

3. It defined an algorithm for finding all twin-primes without requiring extra verification;

4. It completely solved the problem of Twin-Prime Conjecture.

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Table A

k	p_k	p_{k+1}^2-3	$I_k\{p_{k+1}^2-3\}$	$I_k(p_{k+1}^2-3)$	$t_k=\min I_k$
4	7	118	{(11,13),(17,19),··· ,(107,109)}	8	(11,13)
5	11	166	{(17,19),(29,31),··· , (149,151)}	9	(17,19)
6	13	286	{(17,19),(29,31),···, (281,283)}	16	(17,19)
7	17	358	{(29,31),(41,43), ···, (347,349)}	17	(29,31)
8	19	526	{(29,31),(41,43),···, (521,523)}	21	(29,31)
9	23	838	{(29,31),(41,43),···, (827,829)}	29	(29,31)
10	29	958	{(41,43),(59,61),···, (881,883)}	30	(41,43)
11	31	1366	{(41,43),(59,61),···, (1319,1321)}	41	(41,43)
12	37	1678	{(41,43),(59,61),···, (1667,1669)}	48	(41,43)
13	41	1846	{(59,61),(71,73),···, (1787,1789)}	50	(59,61)
14	43	2206	{(59,61),(71,73), ···, (2141,2143)}	61	(59,61)
15	47	2806	{(59,61),(71,73),···, (2801,2803)}	74	(59,61)
16	53	3478	{(59,61),(71,73),···, (3467,3469)}	87	(59,61)
17	59	3718	{(71,73),(101,103),···, (3671,3673)}	91	(71,73)
18	61	4486	{(71,73),(101,103),···, (4481,4483)}	110	(71,73)
19	67	5038	{(71,73),(101,103),···, (5021,5023)}	121	(71,73)
20	71	5326	{(101,103),(107,109),···, (5279,5281)}	123	(101,103)
21	73	6238	{(101,103),(107,109),···, (6197,6199)}	138	(101,103)
22	79	6886	{(101,103),(107,109),···, (6869,6871)}	152	(101,103)
23	83	7918	{(101,103),(107,109),···, (7877,7879)}	166	(101,103)
24	89	9406	{(101,103),(107,109),···, (9341,9343)}	187	(101,103)
25	97	10198	{(101,103),(107,109),···, (10139,10141)}	202	(101,103)
26	101	10606	{(107,109),(137,139), ···, (10529,10531)}	208	(107,109)
27	103	11446	{(107,109),(137,139), ···, (11351,11353)}	218	(107,109)
28	107	11878	{(137,139),(149,151), ···, (11831,11833)}	223	(137,139)
29	109	12766	{(137,139),(149,151), ···, (12611,12613)}	234	(137,139)
30	113	16126	{(137,139),(149,151), ···, (16067,16069)}	276	(137,139)
31	127	17158	{(137,139),(149,151), ···, (17027,17029)}	288	(137,139)
32	131	18766	{(137,139),(149,151), ···, (18539,18541)}	315	(137,139)
33	137	19318	{(149,151),(179,181),···, (19211,19213)}	320	(149,151)
34	139	22198	{(149,151),(179,181), ···, (22157,22159)}	365	(149,151)
35	149	22798	{(179,181),(191,193), ···, (22739,22741)}	374	(179,181)
36	151	24646	{(179,181),(191,193), ···, (24419,24421)}	394	(179,181)
37	157	26566	{(179,181),(191,193), ···, (26261,26263)}	411	(179,181)
38	163	27886	{(179,181),(191,193), ···, (27791,27793)}	432	(179,181)
39	167	29926	{(179,181),(191,193), ···, (29879,29881)}	455	(179,181)
40	173	32038	{(179,181),(191,193), ···, (32027,32029)}	480	(179,181)
41	179	32758	{(191,193),(197,199), ···, (32717,32719)}	492	(191,193)
42	181	36478	{(191,193),(197,199), ···, (36467,36469)}	541	(191,193)
43	191	37246	{(197,199),(227,229), ···, (37199,37201)}	547	(197,199)
44	193	38806	{(197,199),(227,229), ···, (38747,38749)}	567	(197,199)
45	197	39598	{(227,229),(239,241), ···, (39509,39511)}	574	(227,229)
46	199	44518	{(227,229),(239,241), ···, (44381,44383)}	626	(227,229)
47	211	49726	{(227,229),(239,241), ···, (49667,49669)}	685	(227,229)
48	223	51526	{(227,229),(239,241), ···, (51479,51481)}	708	(227,229)
49	227	52438	{(239,241),(269,271), ···, (52361,52363)}	716	(239,241)
50	229	54286	{(239,241),(269,271), ···, (54011,54013)}	732	(239,241)
51	233	57118	{(239,241),(269,271), ···, (56921,56923)}	764	(239,241)
52	239	58078	{(269,271),(281,283), ···, (57899,57901)}	772	(269,271)
53	241	62998	{(269,271),(281,283), ···, (62987,62989)}	818	(269,271)
54	251	66046	{(269,271),(281,283), ···, (65981,65983)}	851	(269,271)
55	257	69166	{(269,271),(281,283), ···, (69149,69151)}	878	(269,271)
56	263	72358	{(269,271),(281,283), ···, (72269,72271)}	921	(269,271)
57	269	73438	{(281,283),(311,313), ···, (73361,73363)}	927	(281,283)
58	271	76726	{(281,283),(311,313), ···, (76649,76651)}	957	(281,283)
59	277	78958	{(281,283),(311,313), ···, (78887,78889)}	977	(281,283)
60	281	80086	{(311,313),(347,349), ···, (79997,79999)}	988	(311,313)

Table B

k	$t_k = \min I_k$	k	$t_k = \min I_k$	k	$t_k = \min I_k$	k	$t_k = \min I_k$
1	(3,5)	206-208	(1289,1291)	460-462	(3299,3301)	681-694	(5231,5233)
2	(5,7)	209-211	(1301,1303)	463-468	(3329,3331)	695-699	(5279,5281)
3-4	(11,13)	212-214	(1319,1321)	469-472	(3359,3361)	700-714	(5417,5419)
5-6	(17,19)	215-224	(1427,1429)	473-474	(3371,3373)	715-718	(5441,5443)
7-9	(29,31)	225-229	(1451,1453)	475-476	(3389,3391)	719-722	(5477,5479)
10-12	(41,43)	230-233	(1481,1483)	477-483	(3461,3463)	723-725	(5501,5503)
13-16	(59,61)	234-235	(1487,1489)	484-485	(3467,3469)	726-728	(5519,5521)
17-19	(71,73)	236-252	(1607,1609)	486-491	(3527,3529)	729-739	(5639,5641)
20-25	(101,103)	253-255	(1619,1621)	492-494	(3539,3541)	740-742	(5651,5653)
26-27	(107,109)	256-261	(1667,1669)	495-497	(3557,3559)	743-744	(5657,5659)
28-32	(137,139)	262-264	(1697,1699)	498-500	(3581,3583)	745-754	(5741,5743)
33-34	(149,151)	265-267	(1721,1723)	501-511	(3671,3673)	755-767	(5849,5851)
35-40	(179,181)	268-276	(1787,1789)	512-523	(3767,3769)	768-771	(5867,5869)
41-42	(191,193)	277-285	(1871,1873)	524-529	(3821,3823)	772-773	(5879,5881)
43-44	(197,199)	286-287	(1877,1879)	530-533	(3851,3853)	774-793	(6089,6091)
45-48	(227,229)	288-293	(1931,1933)	534-541	(3917,3919)	794-798	(6231,6133)
49-51	(239,241)	294-295	(1949,1951)	542-544	(3929,3931)	799-804	(6197,6199)
52-56	(269,271)	296-301	(1997,1999)	545-550	(4001,4003)	805-814	(6269,6271)
57-59	(281,283)	302-306	(2027,2029)	551-554	(4019,4021)	815-818	(6299,6301)
60-63	(311,313)	307-312	(2081,2083)	555-557	(4049,4051)	819-827	(6359,6361)
64-68	(347,349)	313-314	(2087,2089)	558-562	(4091,4093)	828-836	(6449,6451)
69-80	(419,421)	315-317	(2111,2113)	563-566	(4127,4129)	837-845	(6551,6553)
81-82	(431,433)	318-319	(2129,2131)	567-571	(4157,4159)	846-848	(6569,6571)
83-88	(461,463)	320-322	(2141,2143)	572-576	(4217,4219)	849-857	(6659,6661)
89-97	(521,523)	323-331	(2237,2239)	577-578	(4229,4231)	858-861	(6689,6691)
98-103	(569,571)	332-335	(2267,2269)	579-580	(4241,4243)	862-863	(6701,6703)
104-108	(599,601)	336-342	(2309,2311)	581-583	(4259.4261)	864-869	(6761,6763)
109-112	(617,619)	343-345	(2339,2341)	584-585	(4271,4273)	870-871	(6779,6781)
113-115	(641,643)	346-352	(2381,2383)	586-591	(4337,4339)	872-873	(6791,6793)
116-119	(659,661)	353-372	(2549,2551)	592-600	(4421,4423)	874-877	(6827,6829)
120-139	(809,811)	373-376	(2591,2593)	601-607	(4481,4483)	878-883	(6869,6871)
140-141	(821,823)	377-383	(2657,2659)	608-612	(4517,4519)	884-890	(6947,6949)
142-143	(827,829)	384-389	(2687,2689)	613-615	(4547,4549)	891-892	(6959,6961)
144-147	(857,859)	390-394	(2711,2713)	616-624	(4637,4639)	893-912	(7127,7129)
148-151	(881,883)	395-397	(2729,2731)	625-627	(4649,4651)	913-920	(7211,7213)
152-170	(1019,1021)	398-404	(2789,2791)	628-635	(4721,4723)	921-930	(7307,7309)
171-172	(1031,1033)	405-407	(2801,2803)	636-642	(4787,4789)	931-933	(7331,7333)
173-175	(1049,1051)	408-427	(2969,2971)	643-645	(4799,4801)	934-935	(7349,7351)
176-177	(1061,1063)	428-429	(2999,3001)	646-657	(4931,4933)	936-943	(7457,7459)
178-181	(1091,1093)	430-443	(3119,3121)	658-663	(4967,4969)	944-947	(7487,7489)
182-189	(1151,1153)	444-447	(3167,3169)	664-670	(5009,5011)	948-956	(7547,7549)
190-200	(1229,1231)	448-457	(3251,3253)	671-672	(5021,5023)	957-958	(7559,7561)
201-205	(1277,1279)	458-459	(3257,3259)	673-680	(5099,5101)	959-963	(7589,7591)

In table B the notation that $t_k = (11, 13)$ when $k = 3 - 4$ means $t_3 = t_4 = (11, 13)$ and that $t_k = (101, 103)$ when $k = 20 - 25$ means $t_{20} = t_{21} = \cdots = t_{25} = (101, 103)$ etc.

Table C

k	p_{k+1}	s	π_s	p_{k+1}^2-3	$I_k\{\pi_s\}$ (where $p_{k+1} < \pi_s < p_{k+1}^2 - 3 \le \pi_{s+1}$)	$I_k(\pi_s)$	$Av(k,s)$
4	11	3	30	118	{(11,13),(17,19),(29,31)}	3	2.14286
5	13	3	30	166	{(17,19),(29,31)}	**2**	**1.75325**
6	17	4	210	286	{(17,19),(29,31), ⋯, (197,199)}	**12**	**10.3846**
7	19	4	210	358	{(29,31),(41,43), ⋯, (197,199)}	11	9.1629
8	23	4	210	526	{(29,31),(41,43), ⋯, (197,199)}	11	8.19838
9	29	4	210	838	{(29,31),(41,43), ⋯, (197,199)}	11	7.48548
10	31	4	210	958	{(41,43),(59,61), ⋯, (197,199)}	10	6.96924
11	37	4	210	1366	{(41,43),(59,61), ⋯, (197,199)}	10	6.51961
12	41	4	210	1678	{(41,43),(59,61), ⋯, (197,199)}	10	6.1672
13	43	4	210	1846	{(59,61),(71,73), ⋯, (197,199)}	9	5.86636
14	47	4	210	2206	{(59,61),(71,73), ⋯, (197,199)}	**9**	**5.59351**
15	53	5	2310	2806	{(59,61),(71,73), ⋯, (197,199)}	**64**	**58.9103**
16	59	5	2310	3478	{(59,61),(71,73), ⋯, (2309,2311)}	64	56.6873
17	61	5	2310	3718	{(71,73),(101,103), ⋯, (2309,2311)}	63	54.7657
18	67	5	2310	4486	{(71,73),(101,103), ⋯, (2309,2311)}	63	52.9701
19	71	5	2310	5038	{(71,73),(101,103), ⋯, (2309,2311)}	63	51.3889
20	73	5	2310	5326	{(101,103),(107,109), ⋯, (2309,2311)}	62	49.9413
21	79	5	2310	6238	{(101,103),(107,109), ⋯, (2309,2311)}	62	48.5731
22	83	5	2310	6886	{(101,103),(107,109), ⋯, (2309,2311)}	62	47.3434
23	89	5	2310	7918	{(101,103),(107,109), ⋯, (2309,2311)}	62	46.2026
24	97	5	2310	9406	{(101,103),(107,109), ⋯, (2309,2311)}	62	45.1643
25	101	5	2310	10198	{(101,103),(107,109), ⋯, (2309,2311)}	62	44.2331
26	103	5	2310	10606	{(107,109),(137,139), ⋯, (2309,2311)}	61	43.3572
27	107	5	2310	11446	{(107,109),(137,139), ⋯, (2309,2311)}	61	42.5153
28	109	5	2310	11878	{(137,139),(149,151), ⋯, (2309,2311)}	60	41.7206
29	113	5	2310	12766	{(137,139),(149,151), ⋯, (2309,2311)}	60	40.9551
30	127	5	2310	16126	{(137,139),(149,151), ⋯, (2309,2311)}	60	40.2302
31	131	5	2310	17158	{(137,139),(149,151), ⋯, (2309,2311)}	60	39.5967
32	137	5	2310	18766	{(137,139),(149,151), ⋯, (2309,2311)}	60	38.9922
33	139	5	2310	19318	{(149,151),(179,181), ⋯, (2309,2311)}	59	38.4229
34	149	5	2310	22198	{(149,151),(179,181), ⋯, (2309,2311)}	59	37.8701
35	151	5	2310	22798	{(179,181),(191,193), ⋯, (2309,2311)}	58	37.3618
36	157	5	2310	24646	{(179,181),(191,193), ⋯, (2309,2311)}	58	36.8669
37	163	5	2310	26566	{(179,181),(191,193), ⋯, (2309,2311)}	58	36.3973
38	167	5	2310	27886	{(179,181),(191,193), ⋯, (2309,2311)}	58	35.9507
39	173	5	2310	29926	{(179,181),(191,193), ⋯, (2309,2311)}	**58**	**35.5201**
40	179	6	30030	32038	{(179,181),(191,193), ⋯, (2309,2311)}	**456**	**456.423**
41	181	6	30030	32758	{(191,193),(197,199), ⋯, (30011,30013)}	455	451.324
42	191	6	30030	36478	{(191,193),(197,199), ⋯, (30011,30013)}	455	446.337
43	193	6	30030	37246	{(197,199),(227,229), ⋯, (30011,30013)}	454	441.663
44	197	6	30030	38806	{(197,199),(227,229), ⋯, (30011,30013)}	454	437.086
45	199	6	30030	39598	{(227,229),(239,241), ⋯, (30011,30013)}	453	432.649
46	211	6	30030	44518	{(227,229),(239,241), ⋯, (30011,30013)}	453	428.3
47	223	6	30030	49726	{(227,229),(239,241), ⋯, (30011,30013)}	453	424.241
48	227	6	30030	51526	{(227,229),(239,241), ⋯, (30011,30013)}	453	420.436
49	229	6	30030	52438	{(239,241),(269,271), ⋯, (30011,30013)}	452	416.732
50	233	6	30030	54286	{(239,241),(269,271), ⋯, (30011,30013)}	452	413.092
51	239	6	30030	57118	{(239,241),(269,271), ⋯, (30011,30013)}	452	409.546
52	241	6	30030	58078	{(269,271),(281,283), ⋯, (30011,30013)}	451	406.119
53	251	6	30030	62998	{(269,271),(281,283), ⋯, (30011,30013)}	451	402.749
54	257	6	30030	66046	{(269,271),(281,283), ⋯, (30011,30013)}	451	399.539
55	263	6	30030	69166	{(269,271),(281,283), ⋯, (30011,30013)}	451	396.43
56	269	6	30030	72358	{(269,271),(281,283), ⋯, (30011,30013)}	451	393.416
57	271	6	30030	73438	{(281,283),(311,313), ⋯, (30011,30013)}	450	390.491
58	277	6	30030	76726	{(281,283),(311,313), ⋯, (30011,30013)}	450	387.609
59	281	6	30030	78958	{(281,283),(311,313), ⋯, (30011,30013)}	450	384.81
60	283	6	30030	80086	{(311,313),(347,349), ⋯, (30011,30013)}	449	382.071

k	p_{k+1}	s	π_s	$p_{k+1}^2 - 3$	$I_k\{\pi_s\}$ (where $p_{k+1} < \pi_s < p_{k+1}^2 - 3 \leq \pi_{s+1}$)	$I_k(\pi_s)$	$Av(k,s)$
61	293	6	30030	85846	{(311,313),(347,349), ⋯, (30011,30013)}	449	379.371
62	307	6	30030	94246	{(311,313),(347,349), ⋯, (30011,30013)}	449	376.781
63	311	6	30030	96718	{(311,313),(347,349), ⋯, (30011,30013)}	449	374.327
64	313	6	30030	97966	{(347,349),(419,421), ⋯, (30011,30013)}	448	371.92
65	317	6	30030	100486	{(347,349),(419,421), ⋯, (30011,30013)}	448	369.543
66	331	6	30030	109558	{(347,349),(419,421), ⋯, (30011,30013)}	448	367.212
67	337	6	30030	113566	{(347,349),(419,421), ⋯, (30011,30013)}	448	364.993
68	347	6	30030	120406	{(347,349),(419,421), ⋯, (30011,30013)}	448	362.827
69	349	6	30030	121798	{(419,421),(431,433), ⋯, (30011,30013)}	447	360.736
70	353	6	30030	124606	{(419,421),(431,433), ⋯, (30011,30013)}	447	358.668
71	359	6	30030	128878	{(419,421),(431,433), ⋯, (30011,30013)}	447	356.636
72	367	6	30030	134686	{(419,421),(431,433), ⋯, (30011,30013)}	447	354.649
73	373	6	30030	139126	{(419,421),(431,433), ⋯, (30011,30013)}	447	352.717
74	379	6	30030	143638	{(419,421),(431,433), ⋯, (30011,30013)}	447	350.825
75	383	6	30030	146686	{(419,421),(431,433), ⋯, (30011,30013)}	447	348.974
76	389	6	30030	151318	{(419,421),(431,433), ⋯, (30011,30013)}	447	347.152
77	397	6	30030	157606	{(419,421),(431,433), ⋯, (30011,30013)}	447	345.367
78	401	6	30030	160798	{(419,421),(431,433), ⋯, (30011,30013)}	447	343.627
79	409	6	30030	167278	{(419,421),(431,433), ⋯, (30011,30013)}	447	341.913
80	419	6	30030	175558	{(419,421),(431,433), ⋯, (30011,30013)}	447	340.241
81	421	6	30030	177238	{(431,433),(461,463), ⋯, (30011,30013)}	446	338.617
82	431	6	30030	185758	{(431,433),(461,463), ⋯, (30011,30013)}	446	337.009
83	433	6	30030	187486	{(461,463),(521,523), ⋯, (30011,30013)}	445	335.445
84	439	6	30030	192718	{(461,463),(521,523), ⋯, (30011,30013)}	445	333.895
85	443	6	30030	196246	{(461,463),(521,523), ⋯, (30011,30013)}	445	332.374
86	449	6	30030	201598	{(461,463),(521,523), ⋯, (30011,30013)}	445	330.874
87	457	6	30030	208846	{(461,463),(521,523), ⋯, (30011,30013)}	445	329.4
88	461	6	30030	212518	{(461,463),(521,523), ⋯, (30011,30013)}	445	327.958
89	463	6	30030	214366	{(521,523),(569,571), ⋯, (30011,30013)}	444	326.535
90	467	6	30030	218086	{(521,523),(569,571), ⋯, (30011,30013)}	444	325.125
91	479	6	30030	229438	{(521,523),(569,571), ⋯, (30011,30013)}	444	323.732
92	487	6	30030	237166	{(521,523),(569,571), ⋯, (30011,30013)}	444	322.381
93	491	6	30030	241078	{(521,523),(569,571), ⋯, (30011,30013)}	444	321.057
94	499	6	30030	248998	{(521,523),(569,571), ⋯, (30011,30013)}	444	319.749
95	503	6	30030	253006	{(521,523),(569,571), ⋯, (30011,30013)}	444	318.467
96	509	6	30030	259078	{(521,523),(569,571), ⋯, (30011,30013)}	444	317.201
97	521	6	30030	271438	{(521,523),(569,571), ⋯, (30011,30013)}	444	315.955
98	523	6	30030	273526	{(569,571),(599,601), ⋯, (30011,30013)}	443	314.742
99	541	6	30030	292678	{(569,571),(599,601), ⋯, (30011,30013)}	443	313.538
100	547	6	30030	299206	{(569,571),(599,601), ⋯, (30011,30013)}	443	312.379
101-103	–	6	30030	–	{(569,571),(599,601), ⋯, (30011,30013)}	443	–
104-108	–	6	30030	–	{(599,601),(617,619), ⋯, (30011,30013)}	442	–
109-112	–	6	30030	–	{(617,619),(641,643), ⋯, (30011,30013)}	441	–
113-115	–	6	30030	–	{(641,643),(659,661), ⋯, (30011,30013)}	440	–
116-119	–	6	30030	–	{(659,661),(809,811), ⋯, (30011,30013)}	439	–
120-125	–	6	30030	–	{(809,811),(821,823), ⋯, (30011,30013)}	438	–
126	709	6	30030	502678	{(809,811),(821,823), ⋯, (30011,30013)}	**438**	**287.179**
127	719	7	510510	516958	{(809,811),(821,823), ⋯, (510449,510451)}	**4606**	**4868.27**
128-139	–	7	510510	–	{(809,811),(821,823), ⋯, (510449,510451)}	4606	–
140-141	–	7	510510	–	{(821,823),(827,829), ⋯, (510449,510451)}	4605	–
⋮	–	7	510510	–	{(–,–), ⋯, (510449,510451)}	–	–
442	3109	7	510510	9665878	{(3119,3121), ⋯, (510449,510451)}	**4554**	**3273.78**
443	3119	8	9699690	9728158	{(3119,3121), ⋯, (9699647,9699649)}	**57371**	**62161.8**
⋮	–	8	9699690	–	{(–,–), ⋯, (9699647,9699649)}	–	–
1741-1746	–	8	9699690	–	{(15137,15139), ⋯, (9699647,9699649)}	57181	–
1747	14929	8	9699690	222875038	{(15137,15139), ⋯, (9699647,9699649)}	**57181**	**43638.2**
1748	14939	9	223092870	223173718	{(15137,15139), ⋯, (223092671,223092673)}	**895790**	**1003540**